TTIC 31150/CMSC 31150
Mathematical Toolkit (Spring 2023)

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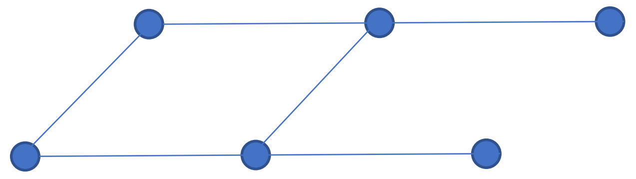
Lecture 16: Random walks on graphs

Recap

- Randomized algorithms for routing to minimize congestion. Randomized complexity classes **RP** and **BPP**, connections to **P/poly**.
- Probability distributions over uncountably infinite spaces, σ-field (σ-algebra), measurability, random variables, CDFs and density functions.
- Gaussian random variables, properties. d-dimensional Gaussians.
- Dimensionality reduction and the Johnson-Lindenstrauss Lemma.
- Tail bounds for sum of independent squared-Gaussian RVs.
- Properties of the unit ball in high dimensions (volume, most points near the surface, most points near the equator, …)

Random Walks on Graphs



Imagine you are lost in a maze. How long will it take you to get out if you just walk around randomly?

General setup: Underlying undirected graph $G = (V, E)$, with n vertices and m edges. Starting from some initial vertex, at each step we move to a random neighbor of current node.

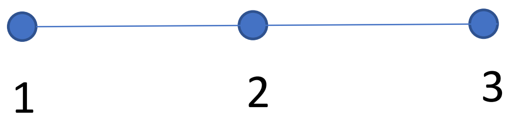
Quantities of interest:

- Hitting time H_{uv}: defined as $\mathbb{E}$[number of steps to reach v | start at u].
- Commute time C_{uv}: $\mathbb{E}$[number of steps to reach v and return to u| start at u].

  Let X_{uv}^{hit} = #steps to reach v starting from u. So, $H_{uv} = \mathbb{E}\left[X_{uv}^{hit}\right]$, $C_{uv} = \mathbb{E}\left[X_{uv}^{hit} + X_{vu}^{hit}\right] = H_{uv} + H_{vu}$.

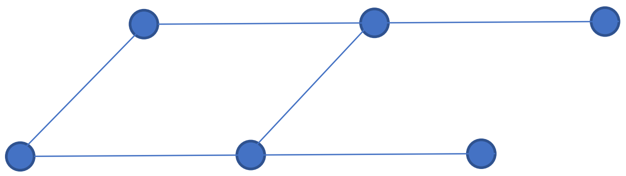
- Cover time from u, Cov_u: $\mathbb{E}$[number of steps to visit all of G| start at u].
- Cover time of G, Cov_G: $\max_u Cov_u$.

Example



H_{12}? H_{21}? H_{31}? Cov_G?

Cover-Time Theorem

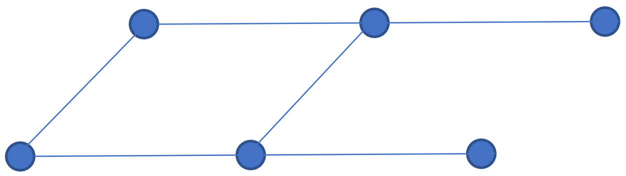


Theorem: If G is a connected graph with n vertices and m edges, then $Cov_G \leq 2m(n-1)$.

So, if you're lost in a maze and walk around randomly, you will visit all the nodes (and hence, the exit) in $O(mn)$ steps.

- On a line, this is tight: it really does take $\Theta(n^2)$ steps in expectation for a random walk to visit all the nodes.
- For some graphs, it is not tight. E.g., for a clique, the cover time is only $O(n \log n)$. Can you see why?
- An example of a graph where cover time is $\Omega(n^3)$ is "lollipop graph": clique of size $n/2$ connected to line of length $n/2$.

Cover-Time Theorem



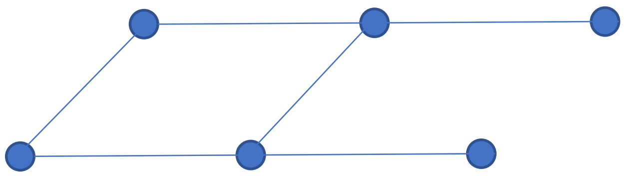
Theorem: If G is a connected graph with n vertices and m edges, then $Cov_G \leq 2m(n-1)$.

For convenience, lets consider current state as being on some edge $\{u, v\}$ headed in some direction (e.g., to v). The theorem will follow from the following key lemma:

Lemma: for any edge/direction (u, v), the expected number of steps between consecutive visits to (u, v) is $2m$.

Note that the lemma implies that if u and v are neighbors, then $C_{vu} \leq 2m$, because expected time $v \rightarrow u \rightarrow v$ is $\leq$ expected time starting from v to take (u, v) edge.

Cover-Time Theorem



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Proof of Theorem from Lemma:

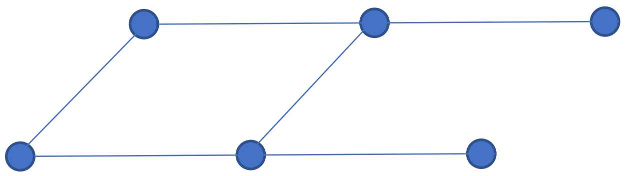
- Consider some spanning tree T of G and some fixed tour of T.

$\mathbb{E}$[time to visit G] $\leq$ $\mathbb{E}$[time to visit nodes in that order]

I.e., E[time from node 1 until visit node 2] + E[time from node 2 until visit node 3] + …

$$= \sum_{\{u,v\} \in T} H_{uv} + H_{vu} \quad = \sum_{\{u,v\} \in T} C_{uv} \quad = 2m(n-1)$$

Proof of key lemma



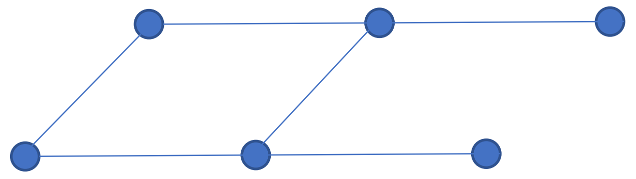
Lemma: for any edge/direction (u, v), the expected number of steps between consecutive visits to (u, v) is $2m$.

First:

- Suppose we started by picking an edge and direction uniformly at random (so our initial distribution has probability $\frac{1}{2m}$ on each directed edge). What does our distribution look like after 1 step?
- Answer: the same. (I.e., this is a stationary distribution)
- For any edge/dir (v, w), $\Pr[\text{on } (v, w) \text{ after 1 step}] = \sum_{u:\{u,v\}\in E} \Pr[\text{was on } (u, v)] \cdot \frac{1}{\deg(v)}$

$$= \frac{\deg(v)}{2m} \cdot \frac{1}{\deg(v)} = \frac{1}{2m}.$$

Proof of key lemma



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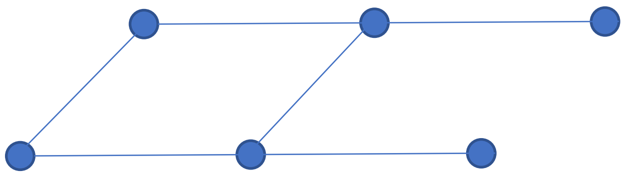
- Answer: the same. (I.e., this is a stationary distribution)
- So, this means that for any directed edge (u, v), in T steps the expected number of traversals of that edge is $\frac{T}{2m}$, by linearity of expectation.

To prove the lemma, we want to invert this, to say that the expected gap between consecutive traversals is $2m$.

Proof of key lemma



Lemma: for any edge/direction (u, v), the expected number of steps between consecutive visits to (u, v) is $2m$.

Note that if our positions at different times t were independent, then this would follow immediately from the fact that the expected value of a Geometric(p) R.V. is $1/p$.

However, these positions are not independent, so we need to be careful. E.g., if the graph consisted of two disconnected pieces with $m/2$ edges each, then the expected time between consecutive traversals would be m but the expected time to the first traversal would be infinite.

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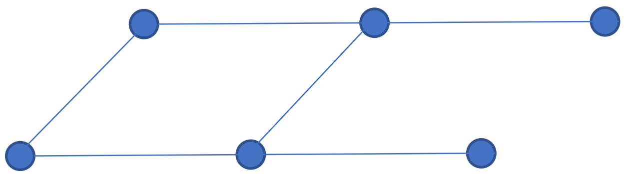
Proof of key lemma



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However, these positions are not independent, so we need to be careful. E.g., if the graph consisted of two disconnected pieces with $m/2$ edges each, then the expected time between consecutive traversals would be m but the expected time to the first traversal would be infinite.

Still, the Geometric RV intuition turns out to be the right one.

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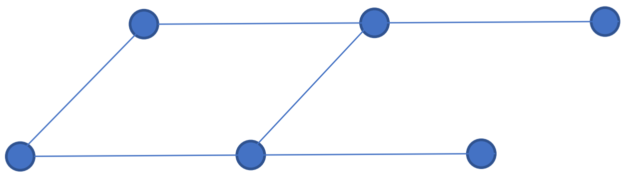
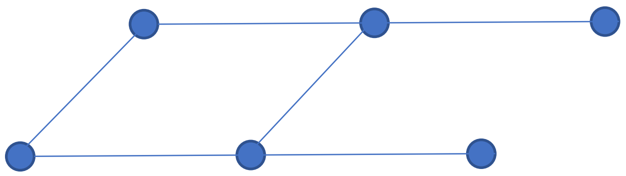
Proof of key lemma



Lemma: for any edge/direction (u, v), the expected number of steps between consecutive visits to (u, v) is $2m$.

- Consider our random walk process starting from the uniform distribution. Let X_1 be an RV denoting the time until we first reach (u, v). Then let X_2 denote the time from that point until our 2nd traversal of (u, v), etc.
- Because the graph is connected, we will indeed reach (u, v) with probability 1.
- In fact, these R.V.s have bounded variance:
  - Wherever you are, there is at least some (perhaps exponentially small) $\delta > 0$ probability that you reach (u, v) in the next n steps.
  - So, our process is dominated by n times a Geometric(δ) RV, which has finite variance.
- As $T \to \infty$, with probability 1 the number of traversals $N \to \infty$ too.

Proof of key lemma



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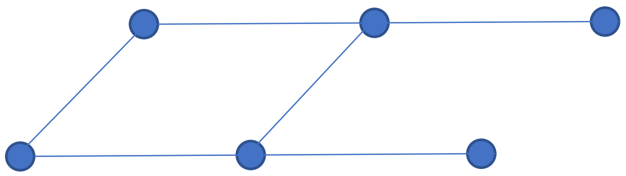
Now, let's apply Chebyshev to $X = \frac{X_1 + \cdots + X_N}{N}$. Let σ^2 be upper bound on $Var[X_i]$.

- Since X_i are independent, $Var[X] \leq \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$. So, $\mathbb{P}[|X - \mathbb{E}[X]| \geq \epsilon] \leq \frac{\sigma^2}{N\epsilon^2}$.
- So, for large N, whp the observed average gap length X is close to its expectation.

$$\ldots\ldots\ldots\ldots\; u - v \ldots\ldots\ldots\ldots\ldots u - v \ldots . u - v \ldots\ldots\ldots\ldots\ldots\ldots u - v \ldots\ldots\ldots\ldots u - v \ldots .$$

Proof of key lemma



Lemma: for any edge/direction (u, v), the expected number of steps between consecutive visits to (u, v) is $2m$.

This means that the fraction of time steps that are traversals of (u, v), namely $1/X$, is also whp multiplicatively close to $1/\mathbb{E}[X]$.

We know the expected fraction of time steps that are traversals is $\frac{1}{2m}$. And if a bounded RV is concentrated, it has to concentrate about its expectation. So, $\mathbb{E}[X] = 2m$.

- Since X_i are independent, $Var[X] \leq \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$. So, $\mathbb{P}[|X - \mathbb{E}[X]| \geq \epsilon] \leq \frac{\sigma^2}{N\epsilon^2}$.
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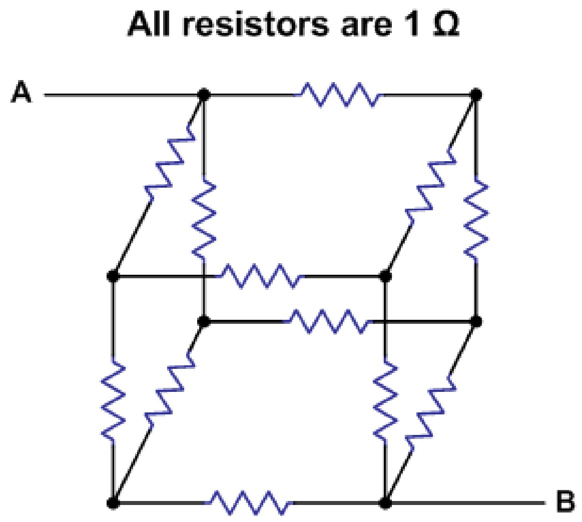
Something completely different(?): electrical networks



Consider a graph G where on each edge we have a resistor of some resistance.

- Say we connect a battery of some voltage V_{batt} between two nodes A and B (so $V_A - V_B = V_{batt}$, and let's for convenience say $V_B = 0$).
- Then each node in the graph will have a voltage (also called "potential") and each edge will have some current flowing in some direction.

Can think of voltage as like "height", and resistors like little water wheels or filters.

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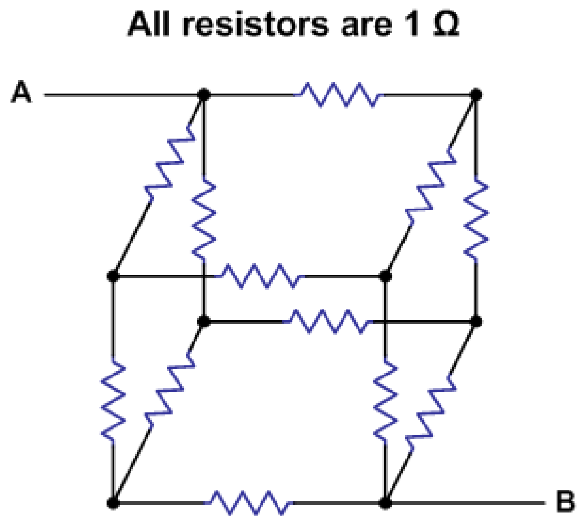


Voltages and currents can be computed using the following two rules.

- Kirchoff's law: current is like water flow: for any node not connected to the battery, flow in = flow out.
- Ohm's law: $V = IR$. Here, R is resistance, V is the voltage drop, and I is the current flow.

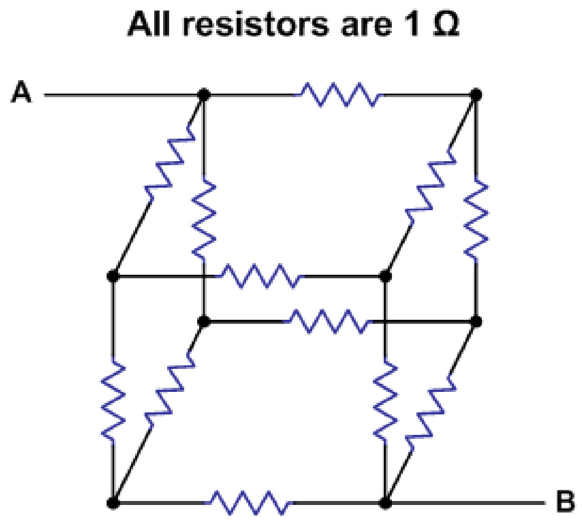
Effective resistance R_{uv} between u and v: connect up battery, measure current, $R_{uv} = \frac{V}{I}$.

Electrical networks and random walks

Consider a graph G, fix two distinguished nodes A,B.

Consider a random walk.

Let p_u be the probability a random walk starting from u reaches A before it reaches B.



Consider placing a 1-volt battery between A and B

Let V_u be the voltage at node u.

Then $p_u = V_u$.

- Solving for p_u: $p_A = 1$, $p_B = 0$, and for all $u \notin \{A, B\}$ we have $p_u = \frac{1}{\deg(u)} \sum_{v:\{u,v\}\in E} p_v$.
- Solving for V_u: $V_A = 1$, $V_B = 0$, and for all $u \notin \{A, B\}$ we have flow in = flow out, which means $\sum_{v:\{u,v\}\in E} \left(\frac{V_v - V_u}{1}\right) = 0$, so $V_u = \frac{1}{\deg(u)} \sum_{v:\{u,v\}\in E} V_v$.

Electrical networks and random walks

Next time: more connections (exact expression for commute time in terms of effective resistance), and rapid mixing.